

Principal Poincaré Pontryagin Function associated to some families of Morse real polynomials

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Abstract

It is known that the Principal Poincaré Pontryagin Function is generically an Abelian integral. We give a sufficient condition on monodromy to ensure that it is an Abelian integral also in non generic cases.

In non generic cases it is an iterated integral. Uribe [17, 18] gives in a special case a precise description of the Principal Poincaré Pontryagin Function, an iterated integral of length at most 2, involving logarithmic functions with only one ramification at a point at infinity. We extend this result to some non isodromic families of real Morse polynomials.

Keywords: Perturbation, First return map, Iterated integrals, Monodromy, Stratification.

MSC: 34M35; 34C08;14D05

1 Introduction

Throughout the paper F denotes a Morse polynomial $F(x, y) : \mathbb{C}^2 \rightarrow \mathbb{C}$ with real coefficients, of degree $d \geq 3$ and with d distinct real points at infinity. It always has $(d-1)^2$ critical points but in non generic cases it can have less than $(d-1)^2$ critical values. The one-form dF defines a foliation of \mathbb{C}^2 . We consider a family of ovals $\delta(t)$ in regular fibers $F = t$ for t in some open interval and a transverse section to these ovals, parametrized by

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t . Let $\omega(x, y)$ be a real polynomial one-form. To a one-parameter foliation defined by the perturbation $dF + \varepsilon\omega$ for a small parameter ε is associated the displacement map which is the difference of the first return map and the identity. It is analytic with respect to ε . The family of ovals is destroyed if and only if the expansion with respect to ε of the displacement map is not identically 0. In order to control the number of isolated zeroes of the displacement map, or in other words the number of limit cycles of the perturbed foliation, it is crucial to know the nature of the first nonzero coefficient of its expansion in ε . It is called the Generating Function in [10]. Following [8] we will call it the Principal Poincaré Pontryagin Function.

It is known that it is an iterated integral [5] and that its length depends on the monodromy group of the Milnor fibration associated to the non perturbed polynomial F [10]. Generically, that is if all $(d-1)^2$ critical values are distinct, this monodromy acts transitively on the homology with complex coefficients of regular fibers $F^{-1}(t)$ and the Principal Poincaré Pontryagin Function is an Abelian integral. It may also be an Abelian integral in non generic cases [12, 13]. If it is not an Abelian integral, the simplest case is the one where it is a length 2 iterated integral, for example if F is a triangle [11], or more generally if F is the product of d linear factors, $F = \ell_1 \cdots \ell_d$, satisfying to the following Hypothesis [17, 18].

Hypothesis 1 *The d points at infinity $\ell_k = 0$ are distinct, all critical points are Morse points, and the 0-level is the only critical level containing more than one critical point. The $d(d-1)/2$ intersection points of the line $\ell_k = 0$ are real.*

These properties ensure that the 0-level of F is what A'Campo calls a divide in [1, 2], see Section 3 for the definition. We keep this terminology.

Definition 1 *A polynomial $F = \ell_1 \cdots \ell_d$ is a generic divide in lines if it satisfies to Hypothesis 1.*

In [18] one of us proves that for generic divides in lines the Principal Poincaré Pontryagin Function is an iterated integral of length at most 2. The proof uses monodromy properties of divides. The divide shows all the homology of regular fibers $F^{-1}(t)$ and allows also to compute the monodromy. Since Hypothesis 1 is stable it is natural to hope that the Principal Poincaré Pontryagin Function remains a length 2 iterated integral after a small perturbation. In Section 2 we give two examples of one-parameter small perturbations of generic divides in lines and we check that the Principal Poincaré Pontryagin

Function is still of length at most 2. Therefore we note that the fibration defined by perturbed polynomial has more monodromy operators than the fibration defined by the generic divide in lines. We also show that if the orbit of some oval generates a codimension 1 subspace of the homology then the Principal Poincaré Pontryagin Function is an Abelian integral.

In Section 3 we generalize examples of Section 2 by introducing

Definition 2 *A connecting family is a continuous family of Morse polynomials $F_\lambda, \lambda \in [0, 1], F_1 = F$ such that all $F_\lambda, \lambda \in]0, 1]$ are isomonodromic and F_0 is a generic divide in lines.*

It is not isomonodromic because F_0 may be more degenerated than F_1 if it has less critical values than F_1 . For regular t we will denote by $H_1^c(t)$ the \mathbb{C} -vector space defined by the homology of the compactification of the fiber $F_\lambda = t, \lambda \in [0, 1]$ with coefficients in \mathbb{C} . We prove following Theorems.

Theorem 1 *Let F_λ be some simple connecting family, $\lambda \in [0, 1]$. Then the \mathbb{C} -vector space generated by the orbit of some oval $\delta(t)$ of the fiber $F_1 = t$ contains $H_1^c(t)$.*

Theorem 2 *If $F_\lambda(x, y)$ is a simple connecting family of Hamiltonians, then for $\lambda \neq 0$ there exist $\nu \leq d$ functions Ψ_1, \dots, Ψ_ν such that each function Ψ_k has logarithmic ramifications at some infinity points of the fibers $F_\lambda^{-1}(t)$ for regular values t and is univalued out of the infinity points, and the Principal Poincaré Pontryagin Function is an element of $\mathbb{C}(t)[x, y, \Psi_1, \dots, \Psi_\nu]$.*

2 Examples

2.1 Perturbations of a product of 3 or 4 linear factors in general position

The polynomial F is a generic divide in lines of degree 3 or 4. We choose some oval $\delta(t)$ for regular t and we denote by $\mathbf{Orb}(\delta(t))$ the vector space generated by the orbit of this oval under the monodromy action.

If $\text{degree}(F)=3$ then for regular t , $\mathbf{Orb}(\delta(t))$ is a 2-dimensional \mathbb{C} -vector space and contains the homology of the compactification of regular fibers, that is $H_1^c(t)$. It is complementary to the \mathbb{C} -vector space generated by two residual cycles at infinity. The coordinates can be chosen in such a way that $F(x, y) = xy(x + y - 1)$ and $F_\varepsilon = (xy + \varepsilon)(x + y - 1)$ (Figure 1). The 0-level

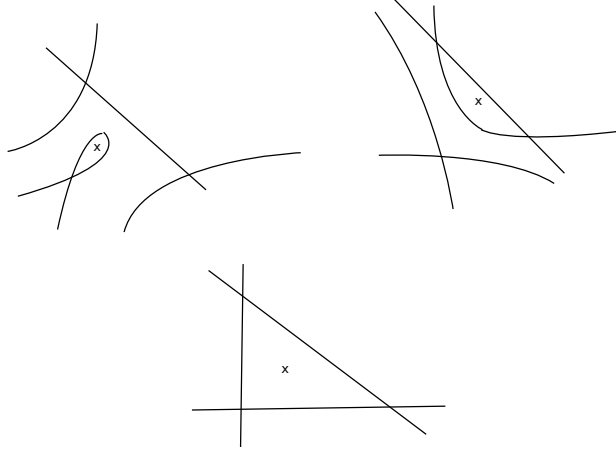


Figure 1: Singular points of the Hamiltonian triangle and its perturbations $(xy + \varepsilon)(x + y - 1)$, for positive values of the parameter on the left and negative values on the right.

and the critical points of F and F_ε are shown in Figure 1. The Principal Poincaré Pontryagin Function is not an Abelian integral [11, 17].

The monodromy group of the Hamiltonian Triangle has only 2 generators: the monodromy around the critical value 0 and the monodromy around the center type critical value. Since Morse points are stable singularities, there remains three saddles and one center after perturbation if ε is sufficiently small but one saddle lies on a nonzero critical level, and now the monodromy group has 3 generators. Up to reparametrization the family F_ε is a simple connecting family, and F_0 is strictly more degenerated than $F_\varepsilon, \varepsilon \neq 0$. One can check that the codimension of $\mathbf{Orb}(\delta(t))$ is 1 in the perturbed case. Then the Principal Poincaré Pontryagin Function is computed using only one multivalued function φ defined by $d\varphi = F_\varepsilon \frac{d(x + y - 1)}{x + y - 1} = (xy + \varepsilon)d(x + y - 1)$ and it is an Abelian integral.

The following perturbation of the generic divide in 4 lines is very similar. Let $F_0 = xy(x + \frac{1}{2}y - 1)(\frac{1}{2}x + y + 1)$. The level $F = 0$ contains 4 lines in general position intersecting at 6 saddle points, the homology of regular fibers is a 9 dimensional \mathbb{C} -vector space, it is generated by three vanishing cycles at center type singular values and the six cycles vanishing at 0, each one surrounding one saddle point. The only critical values are 3 critical values of center type and 0, hence the monodromy group has only 4 generators. The orbit of any oval generates a codimension 3 vector space in $H_1(t)$. Hence one needs 3 functions $\varphi_1, \varphi_2, \varphi_3$ to perform the computation of the Principal Poincaré Pontryagin Function (see [18] for details) and this Principal Poincaré Pontryagin Function is generically not an Abelian integral.

Let now

$$F_\varepsilon = (xy + \varepsilon) \left(x + \frac{1}{2}y - 1 \right) \left(\frac{1}{2}x + y + 1 \right), \varepsilon > 0.$$

In this perturbed situation, the critical level 0 contains 5 saddle points, and the other 4 critical levels contain each one critical point, as can be seen on Figure 2. There are now 5 critical values. Again up to reparametrization the family F_ε is a simple connecting family, and F_0 is strictly more degenerated than $F_\varepsilon, \varepsilon \neq 0$.

The orbit of any oval contains $H_1^c(t)$ and its codimension is 2. Thus one needs now only two functions to compute the Principal Poincaré Pontryagin Function. The Principal Poincaré Pontryagin Function is generically not an Abelian integral, at least if the degree of the perturbative one-form is sufficiently great [18]. We can define φ_1, φ_2 as relative primitives of polynomial one-forms $F_\varepsilon \frac{d(x + \frac{1}{2}y - 1)}{x + \frac{1}{2}y - 1}$ and $F_\varepsilon \frac{d(\frac{1}{2}x + y + 1)}{\frac{1}{2}x + y + 1}$.

We can use an additional parameter to break one more connexion:

$$F_{\varepsilon, \varepsilon'} = (xy + \varepsilon) \left((x + \frac{1}{2}y - 1)(\frac{1}{2}x + y + 1) + \varepsilon' \right), \varepsilon' \ll \varepsilon \ll 1.$$

Now the codimension of $\mathbf{Orb}(\delta(t))$ is one and the Principal Poincaré Pontryagin Function is an Abelian integral. We can put $F_{\varepsilon, \varepsilon'}$ into a chain of two simple connecting families, what we could call a connecting family. Theorem 1 remains true for (non necessary simple) connecting families. Note that $F_{\varepsilon, \varepsilon'}$ is not a two parameter family since we have $\varepsilon' \ll \varepsilon \ll 1$.

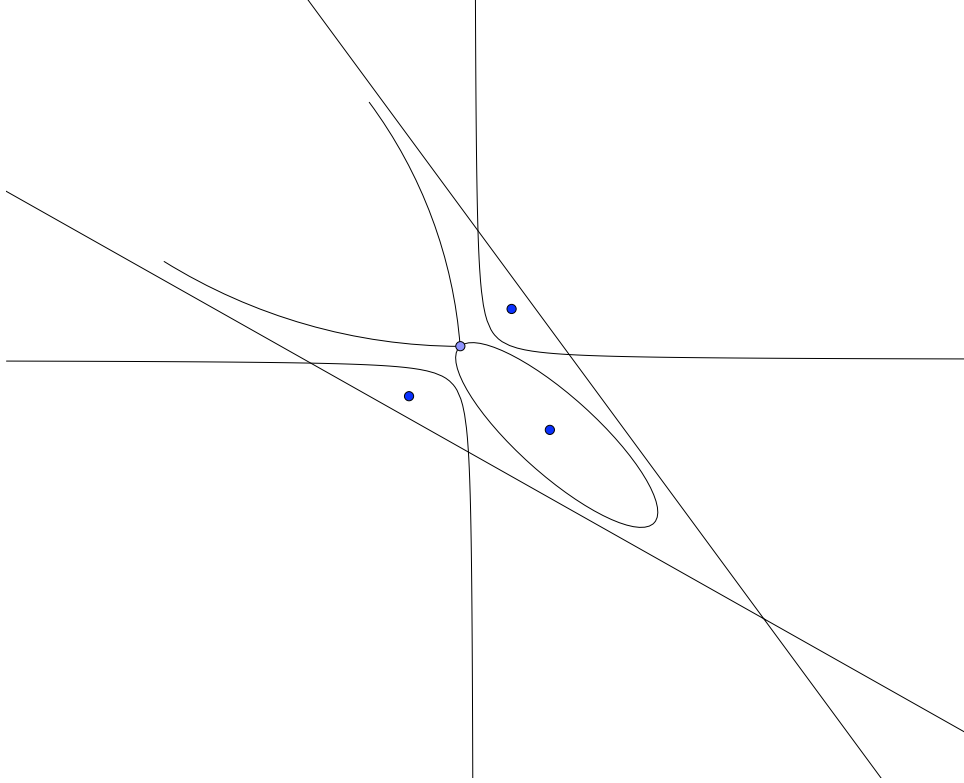


Figure 2: Singular curves in the phase portrait of a perturbation of 4 straight lines. This divide is included in the union of two critical levels.

2.2 Codimension one case

As usually we suppose that F is a generic at infinity real Morse polynomial.

Proposition 1 *If for regular t the codimension of $\mathbf{Orb}(\delta(t))$ is 1 in the \mathbb{C} -vector space $H_1(t)$ then the Principal Poincaré Pontryagin Function is an Abelian integral.*

Proof. We denote by r the dimension of $\mathbf{Orb}(\delta(t))$, so that $r + 1$ is the Milnor number of the fibration defined by F , and we denote by $\delta_1(t), \dots, \delta_r(t)$ a basis of $\mathbf{Orb}(\delta(t))$ for regular t . We complete it to some basis $\delta_1(t), \dots, \delta_r(t), \sigma(t)$ of $H_1(t)$, for regular t . The cycle $\sigma(t)$ can be chosen in such a way that it is invariant under the monodromy action thus it is a residual cycle at

infinity. We use the monodromy representation of the Principal Poincaré Pontryagin Function as defined in [8, 10]. The same letters $\delta_k(t), \sigma(t)$ now denote loops with some base point $p(t)$ or even free loops. Let S the family of loops $\delta_1, \dots, \delta_r$. We construct a family of free loops $\hat{S} = \{hsh^{-1}\}, h \in \Pi_1(F^{-1}(t), p(t)), s \in S$ and finally the main geometric object $H_\delta = \frac{\hat{S}}{[\Pi_1(F^{-1}(t), p(t)), \hat{S}]}$. This means that the elements of H_δ can be uniquely written as $\delta_1^{\alpha_1} \dots \delta_r^{\alpha_r} \cdot \sigma^0$, since we have supposed that $\mathbf{Orb}(\delta(t))$ does not contain σ . Thus there is a natural injection from H_δ into $H_1(F^{-1}(t), \mathbb{Z})$. Moreover the group H_δ is finitely generated and from [8] it has moderate growth. It only remains to use the main result of [10]. \square

This can be applied to the symmetric eight figure of [12, 13] or to the above mentioned example of a perturbation of the Hamiltonian triangle. If $\mathbf{Orb}(\delta(t))$ contains all the homology but a 2-dimensional space generated by 2 residual cycles at infinity, again we denote by r the dimension of $\mathbf{Orb}(\delta(t))$ for regular t , now $r + 2$ is the Milnor number. We complete with two cycles σ_1, σ_2 . Then the elements of H_δ are written as, for instance, $\delta_1^{\alpha_1} \dots \delta_r^{\alpha_r} \cdot \sigma_1^{\beta_1} \cdot \sigma_2^{\beta_2} \cdot \sigma_1^{\gamma_1} \cdot \sigma_2^{\gamma_2}, \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 0$ and to any such free loop there corresponds the same cycle $\alpha_1 \delta_1 + \dots + \alpha_r \delta_r$. Clearly this map is no more injective, the Principal Poincaré Pontryagin Function is an iterated integral on brackets and it is no more an Abelian integral, at least in the general case [11, 17].

We now assume that F is a product of two irreducible polynomials F_a, F_b and that the 0-level $F = 0$ is a divide in the real plane and all the homology of regular fibers $F^{-1}(t)$ is seen on this divide. This is a very particular case, as we will prove.

Lemma 1 *For any oval $\delta(t)$ the orbit $\mathbf{Orb}(\delta(t))$ does not contains all the homology of the fiber $F = t$.*

Proof. The one-form $F_b dF_a$ is relatively cohomologous to $F \frac{dF_a}{F_a}$. Hence its integral is not identically 0 on residual cycles around infinity points $F_a = 0$. So this one-form is not algebraically relatively exact.

Nevertheless $\int_{\delta(t)} F_b dF_a = t \int_{\delta(t)} \frac{dF_a}{F_a} \equiv 0$ since the cycle $\delta(t)$, which is vanishing at some critical value, does not turn around any point at infinity

of fibers $F^{-1}(t)$. \square

Lemma 2 *If the perturbation is $\omega = \alpha F_b dF_a + \beta F_a dF_b$ for some reals α, β , then the family of ovals is not destroyed.*

Proof. First note that $\omega = Fd\left(\ln(F_a^\alpha \cdot F_b^\beta)\right) = Fdg$ with $g = \ln(F_a^\alpha \cdot F_b^\beta)$. The value 0 is a critical one, hence ovals are in some fiber $F = t, t \neq 0$. The only branching points of the logarithm are 0 and ∞ . It is clear that the ovals $\delta(t)$ for regular t don't turn around any branch of $F = 0$, so the function g is univalued along $\delta(t)$. The perturbed polynomial one-form is $dF + \varepsilon Fdg = d(F + \varepsilon Fg) - \varepsilon g dF$. Since g is univalued along $\delta(t)$, $\int_{\delta(t)} d(F + \varepsilon Fg) \equiv 0$ and of course $\int_{\delta(t)} -\varepsilon g dF \equiv 0$, so that $\int_{\delta(t)} dF + \varepsilon \omega \equiv 0$. \square

This is in fact a Darboux integrable case [15]. Following Lemma shows that this case is very particular.

Lemma 3 *The homology is seen on the divide $F_a F_b = 0$ only if the two factors are of degree at most 2 and the degree 2 factors have no real point at infinity. Then the vector space $\mathbf{Orb}(\delta(t))$ contains all the homology H_1^c and its codimension is 1.*

Proof. This result appears in [1], we give a proof for completeness. It is based on the computation of the Euler characteristic since the divide provides a decomposition of the disk. On the divide we see double points and regions. We denote by k the number of saddle points and K the number of compact closed regions of the divide $F = 0$. We suppose that $r_\infty \leq d$ points at infinity of level curves of F are real. Hence this divide has $2r_\infty$ branches going to infinity. Hence

$$1 = K + 2r_\infty - \left(\frac{4k}{2} + 3\frac{2r_\infty}{2}\right) + k + 2r_\infty$$

Thus the homology is seen on the divide if

$$(d-1)^2 = k + K = 1 + 2k - r_\infty.$$

We denote by d_a the degree of F_a , by d_b the degree of F_b . Using $k \leq d_a d_b$ we get

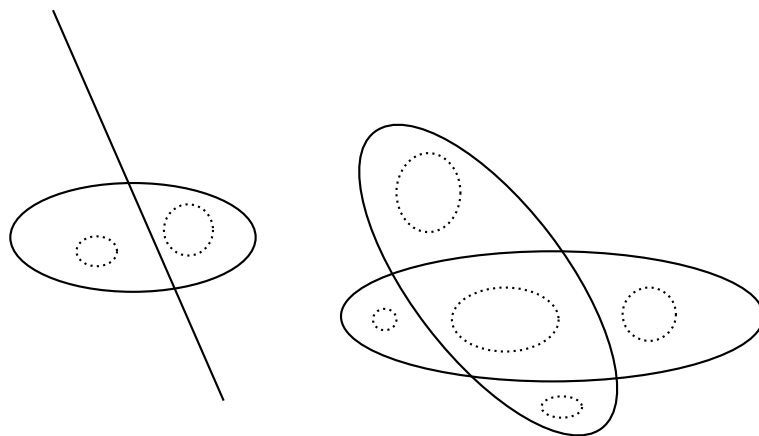


Figure 3: Divide $F = 0$ if $F = F_a F_b$. Dotted curves are ovals. On the left product of a line and a quadratic factor, with 2 critical points of center type and 2 double points. On the right, product of 2 quadratic factors, with 5 critical points of center type and 4 double points.

$$0 \geq -r_\infty \geq d_a(d_a - 2) + d_b(d_b - 2).$$

The first result follows since both degrees d_a, d_b are at least 1. Divides are drawn on Figure 3.

One can check that in the case of a line and an ellipse, $\dim(H_1(t)) = 4$ and $\dim(\mathbf{Orb}(\delta(t))) = 3$. In the case of two ellipses, the Milnor number is 9 and $\dim(\mathbf{Orb}(\delta(t))) = 8$. The cycle which is not in $\mathbf{Orb}(\delta(t))$ has 0 as intersection number with any cycle vanishing cycle at a center critical value, thus it is a residual cycle at infinity. \square

3 Proofs

3.1 Proof of Theorem 1

In this Subsection we prove that if F is in a simple connecting family F_λ , then the vector space $\mathbf{Orb}(\delta(t))$ contains all the homology $H_1^c(t)$ for any oval and regular values t . Therefore we prove that the monodromy group has in some sense more generators in the Milnor fibration defined by F than in the Milnor fibration defined by the divide in lines F_0 .

Lemma 4 *Let $F_\lambda, \lambda \in [0, 1]$ be a simple connecting family of polynomials. The map $(x, y, \lambda) \rightarrow (F_\lambda(x, y), \lambda)$ defines a fibration from $\{(x, y, \lambda)\} = \mathbb{C}^3$ to a subset of $\{(\lambda, t)\} = \mathbb{C}^2$.*

Proof. We consider λ as a complex parameter. From Ehresmann Fibration Theorem [19] this mapping defines a fibration with basis the complement of the set where its rank is not maximal. So the basis of this fibration is the complementary of $\{(c, \lambda)\}$ such that c is a critical value of F_λ . Hence if we denote by Σ_λ the set of critical values of F_λ , the basis of this fibration is $\mathbb{C}^2 \setminus \bigcup \{\lambda \times \Sigma_\lambda\}$. \square

For simplicity we will project the fibers into $\{(x, y)\} = \mathbb{C}^2$ and denote by $F_\lambda^{-1}(t)$ any regular fiber, a Riemann surface in \mathbb{C}^2 . Now again we restrict to $\lambda \in]0, 1]$ where the deformation is isomonodromic. That means the following. From preceding Lemma, if λ and μ are two near values of the parameter, there is a local connection sending Σ_λ to Σ_μ , a regular fiber $F_\lambda^{-1}(t_\lambda)$ to some regular fiber $F_\mu^{-1}(t_\mu)$, and $H_{1,\lambda}(t_\lambda)$ onto $H_{1,\mu}(t_\mu)$. Since

F_λ and F_μ are isomonodromic they have the same number of critical values and the connection establishes a one-one correspondance from Σ_λ onto Σ_μ . Moreover this connection commutes with the monodromy. Namely assume that some critical value c_λ of F_λ is sent to C_μ of F_μ . Then for any cycle $\delta_\lambda(t_\lambda)$ we get the same result if we first let act the monodromy around c_λ and then the connection, or if we first transport the cycle into $H_{1,\mu}(t_\mu)$ and then let act the monodromy around C_μ .

Now the polynomial F_0 is not isomonodromic with F_λ but since all critical points are of Morse type they vary continuously with respect to λ . Hence we have the following:

Lemma 5 *The limit of critical values of F_λ when λ goes to 0 is one of the critical values of F_0 .*

Denote as c_j^λ the critical values of F_λ which go to a critical value of F_0 of center type. Denote as z_j^λ the critical values of F_λ which go to 0 when λ goes to 0.

Lemma 6 *The critical levels c_j^λ contain only one critical point.*

Proof. This is true for $\lambda = 0$. Since it is an open property it is true for λ near 0. By isomonodromy for $\lambda \in]0, 1]$ it is true for $\lambda \in [0, 1]$. \square

The homology of the regular fibers $F_\lambda = t$ vary continuously. The vanishing cycles around critical values c_j^λ will be denoted by δ_j^λ . If a cycle in the Milnor fibration defined by $F_\lambda, \lambda \in]0, 1]$ vanishes at some z_j^λ then when $\lambda \rightarrow 0$ it goes to a vanishing cycle of the divide in lines F_0 , more precisely a cycle vanishing at 0, which shrinks at a double point, intersection of two lines $\ell_m = 0$ and $\ell_n = 0$. We will denote it by $\gamma_{m,n}^\lambda$. The homology at infinity contains all residual cycles around $\ell_n = 0, n = 1, \dots, d$. Moreover we suppose that points at infinity are fixed for $\lambda \in [0, 1]$. With the same notations as above, the connection of Lemma 4 sends the homology at infinity of $F_\lambda^{-1}(t_\lambda)$ onto the homology at infinity of $F_\mu^{-1}(t_\mu)$.

The polynomial $F_0 = \ell_1 \cdots \ell_d$ is a generic divide in lines, thus the critical level $F_0(x, y) = 0$ contains $d(d-1)/2$ saddle points. Since F_0 is a Morse polynomial there are $(d-1)^2$ critical points, hence $K = (d-1)(d-2)/2$ critical points of center type and from genericity hypothesis all these critical points lie on distinct non zero critical levels. The monodromy operators of the degenerated polynomial F_0 are the monodromy around 0 and the

K monodromy operators around critical values of center type, c_j^0 . The monodromy operators of F_λ , $\lambda \in]0, 1]$ are the monodromy around K critical values c_j^λ and monodromy operators around each critical value z_j^λ . We will only use the monodromy generated by a loop turning once counterclockwise around all critical values z_j^λ and only around them, and by K loops turning once counterclockwise around one of the critical values c_j^λ .

Definition 3 *The subgroup of the monodromy generated by the monodromy operators around each c_j^λ and by a loop turning once clockwise around all critical values z_j^λ will be called the sub-monodromy.*

This sub-monodromy of F_λ has $1 + K$ generators, exactly as many generators as the monodromy of the Milnor fibration defined by F_0 . Choose some oval $\delta(t_1)$ in the homology of one regular fiber $F_1 = t_1$. It varies continuously with λ , thus it is in a family denoted by $\delta_\lambda(t_\lambda)$. Its limit when λ goes to 0 is one of the ovals of $F_0 = t_0$ where $[t_0, t_1]$ is a path in \mathbb{C} such that t_λ is regular for F_λ , $\lambda \in [0, 1]$. Since everything depends continuously on λ , the orbit under the action of the sub-monodromy of $\delta(t)$ varies also continuously. Thus the dimension of the vector space generated by this orbit is constant.

Notation 1 *The dimension of $\mathbf{Orb}(\delta_\lambda(t))$ is denoted by r_λ .*

Lemma 7 *The dimension r_λ is at least the dimension in the degenerated case: $r_\lambda \geq r_0$.*

Proof. The vector space $\mathbf{Orb}(\delta_\lambda(t_\lambda))$ contains the vector space generated by the action of the sub-monodromy. \square

When λ goes to 0, the limit of $\mathbf{Orb}(\delta_\lambda(t_\lambda))$ is a vector space containing $\mathbf{Orb}(\delta_0(t_0))$ as a subspace. We know from [18] that $\mathbf{Orb}(\delta_0(t_0))$ contains all homology of the compactification of regular fibers. Moreover the homology at infinity does not depend on λ . That finishes the proof of Theorem 1.

3.2 Proof of Theorem 2

If $\int_{\delta(t)} \omega$ is not identically 0 then it is an Abelian integral and it is the Principal Poincaré Pontryagin Function and we are done. If $\int_{\delta(t)} \omega \equiv 0$ we have to compute further and first we construct a convenient basis of the relative cohomology for some regular t .

Lemma 8 *If the orbit of some cycle $\delta(t)$ is not the whole homology, then there exist polynomial one-forms such that their integral on $\delta(t)$ is identically 0 and their integral on cycles of the complementary of $\mathbf{Orb}(\delta(t))$ is not 0.*

Proof. We denote by r the dimension of $\mathbf{Orb}(\delta(t))$, $\nu = (d-1)^2 - r$. The Petrov module of the integrals of polynomial one-forms on δ has dimension r [9]. It contains the integrals on $\delta(t)$ of polynomial one-forms $\omega_1, \dots, \omega_r$ free as elements of a $\mathbb{C}(t)$ -vector space. This family can be completed with polynomial one-forms to a $\mathbb{C}(t)$ -basis of the relative cohomology. From the dimension of the Petrov module of Abelian integrals on δ , one can construct a basis of the relative cohomology of polynomial one-forms $\omega_1, \dots, \omega_r, \psi_1, \dots, \psi_\nu$ in such a way that the basis is such that $\int_{\delta(t)} \psi_k \equiv 0$ if $k = 1, \dots, \nu$. \square

Remark 1 *Since this is a basis of the relative cohomology the integrals of the one-forms ψ_1, \dots, ψ_ν on cycles of a complementary of the orbit $\mathbf{Orb}(\delta(t))$ are free in the $\mathbb{C}(t)$ -vector space of Abelian integrals, hence also in the $\mathbb{C}[t]$ -module of Abelian integrals.*

Let us use generalized Franoise's algorithm [6, 7] with multivalued functions. If $M_1(t) = -\int_{\delta(t)} \omega \equiv 0$ then there exist polynomials $\alpha_k(F)$, $k = 1, \dots, \nu$ such that the one-form $\omega - \sum_{k=1}^{\nu} \alpha_k(F) \psi_k$ has integral 0 on all cycles of $F = t$, that is this form is topologically relatively exact. From [4] we know that there exists a polynomial T in F such that $T(F) (\omega - \sum_{k=1}^{\nu} \alpha_k(F) \psi_k)$ is algebraically relatively exact, that is there exist polynomials Q, R in x, y such that

$$T(F) \left(\omega - \sum_{k=1}^{\nu} \alpha_k(F) \psi_k \right) = Q(x, y) dF + dR(x, y).$$

This polynomial is called torsion in [4] and it depends on reducible or non connected fibers $F = t$. This yields

$$\omega = \sum_{k=1}^{\nu} \alpha_k(F) \psi_k + \frac{dR(x, y)}{T(F)} + \frac{Q(x, y)}{T(F)} dF.$$

Now $\frac{dR(x, y)}{T(F)} = d \left(\frac{R(x, y)}{T(F)} \right) + T'(F) \frac{R(x, y)}{T^2(F)} dF$ where T' denotes the usual derivative of the polynomial T with respect to F .

We define a primitive of any $\psi_k, k = 1, \dots, \nu$ as follows and we will denote it by Ψ_k . We choose a base point in some regular fiber $F = t_0$, this allows to compute Ψ_k restricted to this fiber. It is multivalued since the one-form ψ_k is not algebraically relatively exact, but it is univalued along $\delta(t_0)$ and along all cycles of $\mathbf{Orb}(\delta(t_0))$. Then we fix a section transversal to our family of regular fibers. This allows to compute Ψ_k on fibers $F = t$ for t in our family of regular values. For any t , the function Ψ_k is not univalued but it is univalued along all cycles of $\mathbf{Orb}(\delta(t))$. Note that

$$\alpha_k(F)\psi_k = d(\alpha_k(F)\Psi_k) - \Psi_k d(F\alpha_k(F)), \quad k = 1, \dots, \nu.$$

Finally there exist functions f_1, g_1 which are polynomials in $x, y, \Psi_1, \dots, \Psi_\nu$ and rational in F such that

$$\omega = g_1(F, x, y, \Psi_1, \dots, \Psi_\nu) dF + df_1(F, x, y, \Psi_1, \dots, \Psi_\nu).$$

And we can go to the next step of the algorithm and compute $M_2(t) = \int_{\delta(t)} g_1 \omega$ that is a length 2 iterated integral. This function lies in the $\mathbb{C}(t)$ -vector space generated by Abelian integrals and integrals such as $\int_{\delta(t)} \Psi_k \psi_j$.

Notation 2 We denote by $I_{k,j} = \int_{\delta(t)} \Psi_k \psi_j, 1 \leq k \leq \nu, 1 \leq j \leq \nu$. If we compute these integrals on the fibers of F_λ we will denote them by $I_k^\lambda(t)$ or $I_{k,j}^\lambda(t)$, and $\psi_k^\lambda, k = 1, \dots, \nu$ the one-forms with relative primitives Ψ_k^λ .

Indeed since $\Psi_k \psi_j + \psi_k \Psi_j = d(\Psi_k \Psi_j)$, integrals $I_{k,j}(t)$ and $I_{j,k}(t)$ are opposite. So we use $I_{k,j}(t), 1 \leq k < j \leq r$. Recall that from the definition of length 2 iterated integrals [5] $I_{k,j}(t) = \int_{\delta(t)} \Psi_k \psi_j$, that is we integrate a multivalued one-form. This makes sense only on paths. So we have to suppose that there is some base point on $F = t$ and that any cycle is represented as a path. By abuse we denote again as $\delta(t)$ this path. If we change the base point the primitive Ψ_k may become $\Psi_k + C_k$ for some constant C_k . We know from [10] that the Principal Poincaré Pontryagin Function is base point independent. Indeed after some change of the base point the length-2 integral $I_{k,j}(t)$ becomes $\int_{\delta(t)} (\Psi_k + C_k) \psi_j$. Its variation is $\int_{\delta(t)} C_k \psi_j$ which is identically 0 by construction. Thus our integrals $I_{k,j}(t)$ are base point independent. It was shown in [11] that they are not Abelian. We show a little more in the following essential Lemma.

Lemma 9 *The integrals of length 2 are free as elements of the $\mathbb{C}(t)$ -vector space generated by Abelian integrals and the $I_{k,j}(t)$.*

Proof. Here we consider $F = F_1$ as an element of a connecting family F_λ . Recall that for $\lambda = 0$, that is for the divide in lines, we have used one-forms $\varphi_k = F \frac{d\ell_k}{\ell_k}, k = 1, \dots, d-1$. We can suppose that the complementary to $\mathbf{Orb}(\delta_\lambda(t))$ contains precisely the residual cycles dual to one-forms $\varphi_k = F \frac{d\ell_k}{\ell_k}, k = 1, \dots, \nu$.

Furthermore the polynomials F_λ have same points at infinity, hence same degree d terms. Hence we can choose one-forms $\psi_k, k = 1, \dots, \nu$ in such a way that for t going to ∞ we have $\psi_k \sim \varphi_k$ for $k = 1, \dots, \nu$. This yields $I_{k,j}^\lambda(t_\lambda) \sim I_{k,j}^0(t_0)$ when t goes to ∞ .

It was proved in [18] that the non Abelian integrals $I_{k,j}^0(t_0)$ are free as elements of a $\mathbb{C}(t)$ -vector space, for $1 \leq k < j \leq d-1$, hence for $1 \leq k < j \leq \nu$. Now we want to know what happens if for some polynomials $\alpha_{k,j}^\lambda(F)$,

$$\sum_{k,j} \alpha_{k,j}^\lambda(t) I_{k,j}^\lambda(t) \equiv 0.$$

If these polynomials are not identically 0 they have higher degree terms. This is the dominating term of preceding combination for t going to ∞ . Since we have supposed that points at infinity of the polynomial F_λ and one forms ψ_k are independent of λ , then this dominating term does not depend on λ . But its limit when λ goes to 0 is 0 because the integrals $I_{k,j,0}^\star(t)$ are free as elements of a $\mathbb{C}(t)$ -vector space. Thus this dominating term is 0 and this is a contradiction. \square

Lemma 10 *The iterated integrals of length 3 such as $\int_{\delta(t)} \psi_k \psi_j \psi_m$ are not base point independent.*

Proof. Again we have to choose a base point and to integrate along paths lying in the fiber $F = t$ and starting at this base point and we denote as $\int_{\delta(t)} \psi_k \psi_j \psi_m$ an iterated integral along some path representing the cycle $\delta(t)$. If we denote by $\gamma(p)$ the piece of δ starting at the base point and going to the point p of δ then $\int_{\delta(t)} \psi_k \psi_j \psi_m$ is computed as the integral along δ of the multivalued one-form which takes the value $\left(\int_{\gamma(p)} \psi_k \psi_j \right) \psi_m(p)$ at p when p varies along δ .

The length 2 iterated integral $\int_{\gamma(p)} \psi_k \psi_j$ is computed as the integral on the path $\gamma(p)$ of the multivalued one-form $\Psi_k \psi_j$. If we move the base point then Ψ_k may become $\Psi_k + C_k$ for some constant C_k , the integral $\int_{\delta(t)} \psi_k \psi_j \psi_m$ becomes $\int_{\delta(t)} \psi_k \psi_j \psi_m + C_k I_{j,m}(t)$. Its variation is nonzero since this integral

$I_{j,m}(t)$ is nonzero. \square

The following result finishes the proof.

Corollary 1 *The only base point independent one-forms are the Abelian integrals and the $I_{j,m}(t)$.*

It was proved in [17] that generically the Principal Poincaré Pontryagin Function of order 2, $M_2(t)$, is not an Abelian integral if the degree of the perturbative one-form is at least 5 and the Hamiltonian F_0 is of degree 3.

4 Conclusion and perspectives

A first generalization would be to use chains of simple connecting families. Namely we can use a simple connecting family connecting F to a more degenerated polynomial which could be less degenerated than a generic divide in lines. The product of these paths in the manifold of Morse polynomials of degree d with fixed d real points at infinity is a connecting family. Proof of Theorem 1 shows that the orbit of some oval in $F = t$ for generic t contains all the homology $H_1^c(t)$. Theorem 2 can be generalized to such families.

Next one could relax hypothesis, for instance allow to points at infinity to move but keep generic at infinity polynomials along all the connecting family. This would allow complex points at infinity, which is natural since the monodromy works in C^2 and thus the fact that points at infinity are real or not is irrelevant. Therefore one has to adapt proof of Lemma 9.

We conjecture that as soon as F is a Morse polynomial of degree d with d distinct points at infinity such that at least $d(d+1)/2$ critical levels contain only one critical point then the Principal Poincaré Pontryagin Function is an iterated integral of length at most 2. This conjecture is based on the stratification of polynomials given by Zariski-Tarskii Theorem (or Chevalley Theorem) [3, 16]. Indeed any Morse polynomial F lies in some stratum. If there is a generic divide in lines in the boundary of this stratum then there exists a simple connecting family from F to the generic divide in lines and we are done. Else we can use the following technic indicated to us by Maxim Kazarian who solves similar questions in [14].

We suppose that F is a real polynomial of degree d with d real points at infinity, that all critical points are of Morse type, that at most $d/2$ critical points lie on the level $F = 0$ and that other critical levels contain only one critical point. We are going to prove that such polynomials can be put in a simple connecting family.

Therefore we choose d lines ℓ_1, \dots, ℓ_d such that the algebraic curve $\ell_1 \cdots \ell_d = 0$ contains all $d/2$ or $(d-1)/2$ critical points of the 0-level of F and both polynomials F and $F_0 = \ell_1 \cdots \ell_d$ have the same points at infinity. Consider the family of polynomials

$$F_\lambda = \lambda F + (1 - \lambda)F_0, \quad F_1 = F, \quad F_0 = \ell_1 \cdots \ell_d.$$

All polynomials $F_\lambda, \lambda \in [0, 1]$ have the same points at infinity. Moreover all $d/2$ or $(d-1)/2$ critical points of the critical level $F = 0$ are critical points of $F_\lambda, \lambda \in [0, 1]$.

If all critical points of F_λ are of Morse type and all non zero critical levels of $F_\lambda, \lambda \in]0, 1]$ contain only one critical point, we have constructed a simple connecting family and the result is proved. Else it means that for some isolated values of λ either one critical point is not Morse or one nonzero critical level contains more than one critical point. By Zariski-Tarski Theorem each of these two conditions define an algebraic set in $\mathbb{C}_{d-1}(x, y)$, vector space of polynomials of degree at most d with fixed points at infinity. Thus it can be avoided by following a path in $\mathbb{C}_{d-1}[x, y]$ instead of $\mathbb{R}_{d-1}[x, y]$. Again we have constructed a simple connecting family and the result is proved.

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